Analytic Description of the Equilibrium Shapes of Elastic Rings Under Uniform Hydrostatic Pressure

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Abstract. The parametric equations of the plane curves determining the equilibrium shapes that a uniform inextensible elastic ring could take subject to a uniform hydrostatic pressure are presented in an explicit analytic form. The determination of the equilibrium shape of such a structure corresponding to a given pressure is reduced to the solution of two transcendental equations. The shapes with points of contact and the corresponding (contact) pressures are determined by the solutions of three transcendental equations. The analytical results presented here confirm many of the previous numerical results on this subject but the results concerning the shapes with lines of contact reported up to now are revised.

Keywords: Elastic ring, hydrostatic pressure, equilibrium shapes, parametric equations
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INTRODUCTION

The present paper addresses the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniformly distributed external force that acts normally to the ring in the ring plane.

Maurice Lévy [22] was the first who stated and studied the problem under consideration and reduced the determination of the foregoing equilibrium shapes in polar coordinates to two elliptic integrals for the arclength and polar angle regarded as functions of the squared radial coordinate. He found also several remarkable properties of the equilibrium ring shapes and concluded that if the pressure p is such that $p < (9/4)(D/\rho^3)$, where D and ρ are the ring bending rigidity and radius of the undeformed shape, respectively, then the ring possesses only the circular equilibrium shape.

Later on, Halphen [17] and Greenhill [15] derived exact solutions to this problem in terms of the Weierstrass elliptic functions on the ground of complicated analyses of the properties of the aforementioned elliptic integrals. Halphen (see [17, p. 235]) found out that non-circular shapes with $n \geq 2$ axes of symmetry are possible only for pressures greater than $p_n = (n^2 - 1)(D/\rho^3)$. Halphen [17] and Greenhill [15] presented also several examples of non-circular equilibrium ring shapes. It should be noted, however, that the exact solutions reported in [17, 15], representing the polar angle as a function of the radius, appeared to be intractable and many researchers continued searching exact solutions [4–10], while others used various approximations [24, 13] on the way

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