Quadratic Casimir Invariants for "Universal" Lie Algebra Extensions

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Abstract. We consider a special kind of Lie-algebra extensions, called "universal" extensions, introduced recently. We investigate the quadratic Casimir invariants for the "universal" extensions, their reductions and extensions.

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INTRODUCTION

In the past decade some developments in the study of Dynamical Systems revealed the importance of the fact that on one and the same vector space one can have different Lie algebra (or Poisson) structures and the most interesting applications arise when these structures are compatible [5, 3, 8]. The above stimulated the search for non-canonical Lie algebra structures and Poisson brackets. The examples when one studies Classical Mecanics systems using Kirillov type Poisson structures (Kirillov tensors [1]) of course are not something surprising today and even such systems as the Kepler dynamics are regarded differently, see for example [7]. However, it is interesting that in other traditional field as Hydrodynamics and Magnetohydrodynamics there are examples and applications of Lie algebras, endowed also with algebraic structure different from the canonical one. Consequently the corresponding Kirillov tensors on the co-algebra spaces are different from the canonical ones [9]. In [9] there has performed the study of a class of such structures. They are characterized by the fact that one starts with arbitrary Lie algebra $\mathscr{G}$ over some fixed field $\mathbb{K}$ and using a special type of $(1,2)$ tensor W_s^{ij} (element from $(\mathbb{K}^n)^* \otimes (\mathbb{K}^n)^* \otimes \mathbb{K}^n$ one builds from it another Lie algebra $\mathscr{G}_W$. More specifically, one takes $\mathscr{G}^n$ and on it defines a new bracket

$$([\mathbf{x},\mathbf{y}]_W)_s = \sum_{i,j=1}^{n} W_s^{ij}[x_i, y_j], \qquad W_s^{ij} \in \mathbb{K}. \tag{1}$$

Here $\mathbf{x},\mathbf{y}$ are elements from $\mathscr{G}^n$ with components $x_i, y_i \in \mathscr{G}$ and brackets $[x_i, y_j]$ are understood as brackets in $\mathscr{G}$. Provided the above is a Lie bracket we obtain the algebra $\mathscr{G}_W$.

In this way one performs a kind of extensions of the original Lie algebra $\mathscr{G}$. As mentioned already these structures have applications in Hydrodynamics and Magnetohydrodynamics but there are examples of finite dimensional system having a Poisson-Lie structure resulting from one of the most simple extensions of that class [10]. The "uni-

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