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THE BRACHISTOCHRONE WITH DRY FRICTION AS THE ISOPERIMETRIC VARIATIONAL PROBLEM

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Abstract. The well-known task on finding the curve of most rapid descent in presence of dry friction is solved as a variational problem.

1. Statement of a Problem

Let Axy be a cartesian frame with axis Ay drawn vertically downwards and origin $A(0,0)$ that is the initial position of a heavy particle of mass m which starts to slide along a plane material curve with dry (Coulomb) friction towards to the final position $B(a,b)$. Put $\mathbf{r}(t) = \{x(t),\ y(t)\}$ the parametric equations of the supporting curve with time t. The equations of motion of the particle are

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg\mathbf{j}\boldsymbol{\tau} - kN, \qquad \frac{mv^2}{\rho} = mg\mathbf{j}\mathbf{n} + N \tag{1}$$

where N is the module of the normal pressure force acting onto the particle from the supporting curve, the coefficient of dry friction is $k < 1$, the magnitude $\rho > 0$ is radius of curvature at the current point $P(x(t),\ y(t))$ of the supporting curve, $\boldsymbol{\tau}$ and $\mathbf{n}$ are the unit vectors correspondingly of the tangent and the normal straight-lines

$$\left(\frac{\dot{x}(t)}{v},\ \frac{\dot{y}(t)}{v}\right), \qquad \left(\frac{\dot{y}(t)}{v},\ \frac{-\dot{x}(t)}{v}\right), \qquad v = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\,.$$

We denote the unit vector of the descending vertical by $\mathbf{j}$ and the acceleration of a free fall by g. Let the supporting curve be convex downwards.