ON A FULL QUANTIZATION OF THE TORUS

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Abstract

I exhibit a prequantization of the torus which is actually a "full" quantization in the sense that a certain minimal complete set of classical observables is irreducibly represented. Thus in this instance there is no Groenewold-Van Hove obstruction to quantization.

1. INTRODUCTION

The prequantization procedure produces a faithful representation of the entire Poisson algebra of a quantizable symplectic manifold.[1] In general, these prequantization representations are flawed physically; for instance, the prequantization of $\mathbf{R}^{2n}$ with its standard symplectic structure is not unitarily equivalent to the Schrödinger representation. One usually remedies this by imposing an irreducibility requirement. But there is seemingly a price to be paid for irreducibility: one can no longer quantize *all* classical observables, but rather only proper subalgebras thereof.

This "obstruction" to quantization has been known since the 1940s. In a series of papers, Groenewold and later Van Hove showed that it is impossible to quantize the entire Poisson algebra of polynomials on $\mathbf{R}^{2n}$ in such a way that the Heisenberg $\mathrm{h}(2n)$ subalgebra of inhomogeneous linear polynomials is irreducibly represented.[2–4] Some of the maximal subalgebras of polynomials that *can* be consistently quantized subject to this irreducibility requirement are the inhomogeneous quadratic polynomials, and polynomials which are at most affine in the momenta or the configurations. See Refs. 5-7 for further discussions of this example. Recently, a similar phenomenon was observed for S^2. In this case it was shown that the maximal subalgebra of the Poisson algebra of spherical harmonics that can be consistently quantized while irreducibly representing the $\mathrm{u}(2)$ subalgebra of spherical harmonics of degree at most one is just this $\mathrm{u}(2)$ subalgebra itself.[8]

Based on these results as well as general quantization theory, it would seem reasonable to conjecture that a "no-go" theorem must always hold, to wit:

> *It is impossible to quantize the entire Poisson algebra of any symplectic manifold subject to an irreducibility requirement.*