QUANTIZATION BY QUADRATIC POLYNOMIALS IN CREATION AND ANNIHILATION OPERATORS

Wojtek Słowikowski

Institute of Mathematics, Aarhus University
Ny Munkegade, 8000 Aarhus C
Denmark

Let us fix a number $q = 1$ or -1. To a given Hilbert space $\mathcal{H}, <,>$, we attach an algebra $\Gamma_0\mathcal{H}$ generated by $\mathcal{H}$ and a unity $\emptyset$ called the *vacuum*. We call $\Gamma_0\mathcal{H}$ a *Fock algebra* if the scalar product from $\mathcal{H}$ is extended over $\Gamma_0\mathcal{H}$ in such a way that $< \emptyset, \emptyset >= 1$ and, for every $x \in \mathcal{H}$, the operator $a^+(x)$ of multiplication by x admits the adjoint $a(x)$, defined on the whole $\Gamma_0\mathcal{H}$ and annihilating the vacuum, i.e. $< xf, g >=< f, a(x)g >$ for all $f, g \in \Gamma_0\mathcal{H}$ and $a(x)\emptyset = 0$. We assume that $a(x)$ fulfills the q-Leibnitz rule, i.e. $[a(x), a^+(y)]_q =< x, y > I$, where $[A, B]_q = AB - qBA$.

In the case $q = 1$, the algebra $\Gamma_0\mathcal{H}$ is commutative and is called a Bose algebra, whereas the case $q = -1$ makes the generators from $\mathcal{H}$ anticommute and $\Gamma_0\mathcal{H}$ is then called a Fermi algebra.

We denote by $\Gamma\mathcal{H}$ the completion of $\Gamma_0\mathcal{H}, <,>$ and we write $\mathcal{H}^n$ for the closure of the linear span of $\{x_1 \cdots x_n : x_1, ..., x_n \in \mathcal{H}\}$.

Take an orthonormal basis $\{e_n\}$ in $\mathcal{H}$. To each operator $A \in \mathcal{B}(\mathcal{H})$, we assign an operator

$$d\Gamma A = \Sigma_{j=1}^{\infty} a^+(Ae_n)a(e_n),$$

which is the unique extension of A to a derivation in $\Gamma_0\mathcal{H}$, and hence does not depend on the choice of $\{e_n\}$. For $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$[d\Gamma A, d\Gamma B] = d\Gamma[A, B],$$

i.e. the transformation $A \to d\Gamma A$ is a Lie algebra homomorphism.

Denote by $\mathcal{L}_{hs}^q(\mathcal{H})$ the space of all Hilbert-Schmidt conjugate linear operators $L : \mathcal{H} \to \mathcal{H}$ such that $L' = qL$, where L' denotes the real adjoint to L.

We define the quadratic polynomial

$$h_L = \Sigma_{j=1}^{\infty} e_n(Le_n) \in \mathcal{H}^2$$

and observe that, for $K, L \in \mathcal{L}_{hs}^q(\mathcal{H})$,

$$< h_L, h_K >= 2q \text{ tr } KL.$$

Then, to each $L \in \mathcal{L}_{hs}^q(\mathcal{H})$, we assign the operator

$$a^+(h_L) : \Gamma_0\mathcal{H} \to \Gamma\mathcal{H}$$

of multiplication by h_L. The adjoint $a(h_L)$ of $a^+(h_L)$ is well defined on $\Gamma_0\mathcal{H}$.