NONCOMMUTATIVE SPACE-TIME IMPLIED BY SPIN

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Abstract

This is a short report on our recent study[1] of extended phase spaces for spinning particles. We emphasize the intriguing noncommutativity of space-time, arising by passing from canonical to 'covariant' position variables in our model.

1. INTRODUCTION

Souriau[2] has defined (relativistic) *elementary systems* as mechanical systems whose space of 'motions' ('l'espace des mouvements') is a symplectic manifold with a transitive symplectic action of the Poincaré group (Wigner's philosophy for classical particles). By the momentum mapping theory, transitive actions correspond to coadjoint orbits of the Poincaré group.

We propose[1] a similar algorithm to construct *extended phase spaces* in which motions are solutions of equations of motion, not only abstract points of coadjoint orbits. Like the 'space of motions', an extended phase space is defined as a symplectic 'transitive' space, the transitivity this time being understood with respect to the pair *group* + *space-time* rather than to the group alone (we 'represent' not only the infinitesimal generators of the group but also functions on space-time, to deal directly with space-time localization).

The simplest example of an extended phase space is provided by the cotangent bundle T^*M to the Minkowski space-time M, together with the action of the Poincaré group G, the (connected) group of affine transformations of M leaving invariant the Lorentz metric g. The action of G on T^*M, implied by the action of G on M, has a canonical momentum mapping $J: T^*M \to \mathfrak{g}^*$ ($\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G). Taking inverse images of coadjoint orbits by J leads to a decomposition of T^*M into coisotropic submanifolds (mass shells), whose characteristic foliation determines the (phase) trajectories of the elementary system with a given mass and without spin. T^*M is therefore an extended phase space for massive spinless particles. In order to characterize general extended spaces, we extract three essential properties of the above example (we set $P := T^*M$):

1. P is a Hamiltonian G-space, in other words,

$$\boxed{\text{a complete Poisson map } J: P \to \mathfrak{g}^* \text{ is given}} \tag{1.1}$$