SYMMETRY GROUPS OF THE MIC-KEPLER PROBLEM AND THEIR UNITARY REPRESENTATIONS

TOSHIHIRO IWAI

Department of Applied Mathematics and Physics, Kyoto University
Kyoto 606-01, Japan

and

YOSHIO UWANO

Department of Applied Mathematics and Physics, Kyoto University
Kyoto 606-01, Japan

ABSTRACT

It is well known both in classical and quantum mechanics that the Kepler problem (or the hydrogen atom) admits the symmetry groups, $SO(4)$, $E(3)$, or $SO^+(1,3)$, according as the energy is negative, zero, or positive. However, only part of the unitary irreducible representations are realized as the symmetry group for the Kepler problem.[1] A question now arises: Are the other unitary irreducible representations realizable as symmetry groups for a "modified" Kepler problem ?

This question is worked out in this article. Both in classical and quantum mechanics, the Kepler problem is generalized to the MIC-Kepler problem. It will be shown that the quantized MIC-Kepler problem carries almost all the unitary irreducible representations of $SU(2) \times SU(2)$, $SU(2) \otimes_s \mathbf{R}^3$, or $SL(2, \mathbf{C})$, according as the energy is negative, zero, or positive, which groups are the double covers of $SO(4)$, $E(3)$, and $SO^+(1,3)$, respectively.

1. Setting up the MIC-Kepler problem

The MIC-Kepler problem is to be defined as a reduced system of the conform Kepler problem defined on $T^*(\mathbf{R}^4 - \{0\})$. The key to the reduction is the princip $U(1)$ bundle $\pi: \mathbf{R}^4 - \{0\} \to \mathbf{R}^3 - \{0\}$, the $U(1)$ action ϕ_t and the projection π which are given, respectively, by

$$\phi_t : q \longmapsto T(t)q$$

with

$$T(t) = \begin{pmatrix} N(t) & 0 \\ 0 & N(t) \end{pmatrix}, \quad N(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix},$$

and

$$\pi(q) = (2(q_1q_3 + q_2q_4),\ 2(-q_1q_4 + q_2q_3),\ q_1^2 + q_2^2 - q_3^2 - q_4^2),$$