ON THE CONNECTION BETWEEN ORBITS AND REPRESENTATIONS

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1. Introduction

The title alludes to the correspondence between the coadjoint orbits of connected Lie groups and the representations of such groups; such a correspondence has been established for large classes of groups, e.g. nilpotent and more generally solvable Lie groups, for compact Lie groups and to a large extent for semi-simple Lie groups, and also for more general Lie groups. The complex of ideas connected with this correspondence is sometimes referred to as 'the orbit method'. A few references of an expository nature are [17,18,19].

Here we shall present some results about nilpotent Lie groups related to the orbit method: these results can be viewed on one hand as a generalization of basic results in representation theory of compact groups (or more generally of results concerning representations of a locally compact group which are square integrable modulo the projective kernel), and one the other hand as a generalization of the so called Weyl quantization, which is a quantization procedure for ordinary phase space, and which can be viewed naturally in the context of the simplest non-ablian nilpotent Lie group, the Heisenberg group.

2. Compact groups

In order to prepare for our description of matrix coefficients of nilpotent Lie groups in Section 7 we shall here briefly present a few facts from the representation theory of compact groups.

Let G be a compact group, and let π be a continuous, unitary representation of G on a Hilbert space $\mathcal{H}$ ($\dim \mathcal{H} < +\infty$). Denoting by $\mathcal{B}(\mathcal{H})$ the space of linear maps from $\mathcal{H}$ into itself we define for each $A \in \mathcal{B}(\mathcal{H})$ the function f_π^A on G by

$$f_\pi^A(s) = \mathrm{Tr}(\pi(s)A), \quad s \in G,$$

and set $\mathcal{R}(G,\pi)$ to be the set of such functions f_π^A. The space $\mathcal{R}(G,\pi)$ is the set of matrix coefficients of π, i.e., the linear span of functions of the form $s \to (\pi(s)\xi|\eta)$, $s \in G$, $\xi, \eta \in \mathcal{H}$. The following are basic results in representation theory of compact groups:

(i)

$$\mathcal{R}(G,\pi) \perp \mathcal{R}(G,\rho)$$

if π and ρ are non-equivalent irreducible representations of G (here orthogonality is inside the space $L^2(G)$ defined by Haar measure ds);

(ii)

$$\int_G f_\pi^A(s)\overline{f_\pi^B(s)}ds = \frac{1}{\dim \mathcal{H}}\mathrm{Tr}(AB^*)$$

for each irreducible representation π of G and all $A, B \in \mathcal{B}(\mathcal{H})$ (we assume that the total volume of Haar measure is equal to 1);

(iii)

$$\int_G f_\pi^A(s^{-1})\pi(s)ds = \frac{1}{\dim \mathcal{H}}A$$

for all $A \in \mathcal{B}(\mathcal{H})$.

These results can rather easily be generalized to square integrable representations of locally compact groups, and more generally to representations which are square integrable modulo the