ON THE SPECTRUM OF THE GEODESIC FLOW ON SPHERES

Ivailo M. Mladenov[1] and Vasil V. Tsanov[2]

[1] Central Laboratory of Biophysics, Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl.21, 1113 Sofia, Bulgaria
[2] Faculty of Mathematics and Informatics, Sofia University
1126 Sofia, Bulgaria, E-mail tsanov@bgearn.bitnet

Abstract

We propose a uniform method for derivation of the energy spectrum of the geodesic flow of the sphere S^n (and hence of the Kepler problem) for all dimensions $n \geq 1$. The idea is to use Marsden-Weinstein reduction in the context of equivariant cohomology. The one-dimensional case is thus covered by the general geometric quantization scheme.

In the present note we propose a general procedure which produces the spectrum (with the multiplicities) of the geodesic flow on the n-dimensional sphere. In view of previous work of many authors,[1-6] the point is to include in the "geometric" quantization scheme[7] the case $n = 1$. We recall that this problem is equivalent with the problem of quantization of the n-dimensional hydrogen atom. Because of limited space we do not reproduce here all proofs and computations, which shall be given elsewhere. The authors are convinced, that the trick introduced (using equivariant instead of ordinary cohomology) should work in several other important cases, and see the treatment of the geodesic flow of S^n bellow as a useful example.

The geodesic flow on S^n is the Hamiltonian system (P, σ, F) where,

$$\begin{aligned} &P = T^*S^n = \{(\xi, \eta) \in R^{n+1} \times R^{n+1};\ |\xi| = 1,\ \langle \xi, \eta \rangle = 0\} \\ &\sigma = d\eta \wedge d\xi, \quad \Phi = |\xi|^2 |\eta|^2/2, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{1}$$

The orbits of these Hamiltonian systems are the great circles on the respective spheres. The energy hypersurfaces $\Phi = \epsilon$ (fixed velocities) can be easily identified with the Stiefel manifolds of oriented orthonormal two-frames in R^{n+1},

$$V(2, n+1) = SO(n+1)/SO(n-1),$$

thus

$$P_{1/2} = \Phi^{-1}(1/2) = V(2, n+1)$$

and

$$P_\epsilon \cong P_{1/2} \qquad (\epsilon \neq 0)$$

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by rescaling

$$\eta \to \sqrt{2\epsilon}\eta.$$

Using stereographic projection the sphere S^n is mapped onto R^n and these mappings can be further extended to symplectomorphisms between the corresponding phase spaces T^*S^n and T^*R^n. Besides, these symplectomorphisms map the Hamiltonian of the respective Kepler problem (M, ω, H) in R^n,

$$M = R^{2n}, \quad \omega = dp \wedge dq, \qquad (q,p) \in R^n \backslash \{0\} \times R^n,$$

$$H = p^2/2 - 1/|q|, \qquad |q|^2 = q_1^2 + q_2^2 + \ldots + q_n^2$$

onto the kinetic energy function on T^*S^n and send the (regularized) Kepler motion in R^n onto the geodesic flow on S^n. The corresponding energy hypersurfaces P_ϵ are mapped diffeomorphically onto

$$M_E = \{(q,p) \in M;\ H(q,p) = E\},$$

where $E = -\frac{1}{4\epsilon}$. The orbits that lie on these energy hypersurfaces are parametrized by the points of the Grasmannians

$$G(2, n+1) = SO(n+1)/(SO(n-1) \times SO(2))$$

of oriented two-planes in R^{n+1}. These Grasmannians are compact Hermitean symmetric spaces which are isometric to the nonsingular $(n-1)$-dimensional complex quadrics

$$Q_{n-1} = \{[z_1, z_2, \ldots, z_{n+1}] \in CP^n;\ \sum_{j=1}^{n+1} z_j^2 = 0\}$$

equipped with the canonical Kaehler structure induced by the Fubini-Study metric in CP^n.

The cases $n = 1, 2$ and 3 deserve special considerations. E.g., when $n = 3$, we have the standard Kepler problem in R^3 and Q_2 is a ruled complex surface which is a product of two copies of CP^1. Quantization of the above manifold was done by Simms.[8]

If $n = 2$, we get Q_1 which is isomorphic to CP^1. In this case the Stiefel manifold $V(2,3)$ of orthonormal two-frames in R^3 is isomorphic to the Lie group SO(3), which is non-simply connected, i.e.,

$$\pi_1(SO(3)) = H_1(SO(3), Z) = Z_2 \neq 0.$$

The problems occurring in this situation are treated in more details in our previous work.[9] The correct energy levels and multiplicities can be obtained if one takes into account only the line bundles on the reduced phase space which are restrictions of quantum line bundles on CP^2.

The innovation proposed here is to use $SO(n+1)$ equivariant, rather than ordinary cohomology on the orbit space, in order to obtain a uniform solution of the problem for all dimensions, including $n = 1$. In the present note we just sketch the scheme, by showing that the old and the new scheme give the same result when $n > 1$ and compute the spectrum in the case $n = 1$ to obtain the classical results.[2,5]

As there are several expositions[10,11] of the fundamental results of equivariant cohomology, we just state the relevant definitions and results, by using the notation of Atiyah and Bott.[12]

Let M be a manifold and let G be a Lie group acting on M. Let EG and $BG = EG/G$ be respectively the universal principal bundle and the classifying space for the group G. We denote by M_G the associated M-bundle

$$M_G = M \times_G EG.$$

Then the equivariant cohomology ring with coefficients in the ring F is defined by

$$H_G^*(M, F) \cong H^*(M_G, F). \tag{2}$$

When $K \subset G$ is a Lie subgroup and M is the homogeneous space G/K, we have

$$H_G^*(M, F) \cong H^*(EG/K, F) = H^*(BK, F).$$

In particular

$$H_G^*(pt, F) \cong H^*(BG, F). \tag{3}$$

It is well known that if K is a torus of dimension k the above cohomology ring is just the (cut) polynomial ring of k generators of degree two with coefficients in F , i.e.

$$H^*(BK, F) \cong F(u_1, \ldots, u_k).$$

If G is a compact Lie group with maximal torus K and Weyl group W then we have

$$H^*(BG, F) \cong H^*(BK, F)^W \cong F(u_1, \ldots, u_k)^W. \tag{4}$$

i.e. the cohomology ring of the classifying space BK of the group K consists of the W-symmetric polynomials and is again generated by k elements of even degree (the "elementary symmetric functions"). In any case the equivariant cohomology ring $H^*(BG, Z) = H_G^*$ labels the irreducible representations of the group G.

We shall always interpret $H_G^*(M, R)$ as the equivariant de Rham cohomology ring of M as described in Ref. 12. Let (M, σ) be a symplectic manifold with a G-invariant symplectic form σ, and let

$$J : M \to \mathbf{g}^* \tag{5}$$

be a moment map for the Hamiltonian action of G on M. Then the map J determines a unique "equivariant extension"

$$\sigma \to \sigma^\# \in H_G^*(M, R)$$

(see Ref. 12, Prop. 6.18).

Now the $SO(2)$ action defined by the geodesic flow of the sphere on the symplectic manifold (P, σ) with a momentum map

$$\Phi : \ P \to R \tag{6}$$

described in formula (1), commutes with the natural symplectic action of $SO(n+1)$ on the same manifold (we take the obvious action of $SO(n+1)$ on S^n and lift it to the cotangent bundle). In our previous treatment of the geodesic flow[9,13] for $n > 1$ we have reduced the symplectic form σ to a form σ_ϵ on the orbit space $\Phi^{-1}(\epsilon)/SO(2) \cong Q_{n-1}$ - the nonsingular quadric in P^n, via the Marsden-Weinstein reduction theorem[14]. Then using the geometric quantization integrality condition of Czyz[15] and Hess[16] on the cohomology class

$$\sigma_\epsilon - \frac{1}{2}c_1(Q_{n-1}) \in H^2(Q_{n-1}, Z)$$

we have obtained the spectrum of the problem, i.e. the admissible values ϵ_N and their multiplicities m_N as

$$m_N = \dim H^0(Q_{n-1}, L_N),$$

where L_N is the holomorphic line bundle with

$$c_1(L_N) = \sigma_{\epsilon_N} - \frac{1}{2}c_1(Q_{n-1}).$$

This procedure (initiated by Simms[8]) obviously does not work when $n = 1$ because Q_0 is just the disjoint union of two points ($\epsilon \neq 0$).

We identify $so(n+1)^*$ (via the Killing form) as the space of all antisymmetric matrices with the (co)adjoint action of $SO(n+1)$. The moment map

$$J : P \longrightarrow so(n+1)$$

of the natural action of $SO(n+1)$ on $P = T^*S^n$ is given by

$$J_{ij}(\xi, \eta) = \eta_i \xi_j - \eta_j \xi_i, \quad i, j = 1, ..., n+1.$$

Obviously

$$\{\Phi, J_{ij}\} = 0$$

for all i, j, because the Hamiltonian Φ is invariant with respect to the action of $SO(n+1)$.

Thus the equivariant extension $\sigma^\#$ of σ is invariant under the $SO(2)$ action defined by (P, σ, Φ) (the geodesic flow). This allows us to "reduce" $\sigma^\# \in H^*_{SO(n+1)}(P, R)$ to an element

$$\sigma^\#_\epsilon \in H^*_{SO(n+1)}(Q_{n-1}, R) = H^*(B(SO(n-1) \times SO(2)), R).$$

The admissibility condition for the parameter ϵ is now the condition

$$\sigma^\#_\epsilon - (1/2)c_1(Q_{n-1})^\# \in H^*_{SO(n+1)}(Q_{n-1}, Z) = H^*(B(SO(n-1) \times SO(2)), Z) \tag{7}$$

which gives the spectrum:

$$\epsilon_N = \frac{1}{2}(N + \frac{n-3}{2})^2, \qquad N = 1, 2, \ldots \tag{8}$$

The multiplicities m_N of the spectral values ϵ_N are the dimensions of the corresponding representations:

$$m_N = \frac{2N+n-3}{N+n-2}\binom{N+n-2}{N-1}$$

valid for all values of n and N, except $n = N = 1$ (see bellow).

Now let $n = 1$. Then (see Ref. 12)

$$\sigma^{\#} = \sigma - Ju, \tag{9}$$

where u is the generator of $H^*_{SO(2)}(pt, R) = H^*(BSO(2), R)$ and J is as in (5) the moment map

$$J(\xi, \eta) = \xi_1\eta_2 - \xi_2\eta_1.$$

One computes easily that

$$J^2 = 2\Phi.$$

Now if we reduce the element (9) at $\Phi = \epsilon$, the condition (7) becomes

$$\pm\sqrt{2\epsilon}u = \sigma^{\#}_{\epsilon} \in H^*(BSO(2), Z)$$

whence

$$\pm\sqrt{2\epsilon}u = lu$$

for some integer l.

The admissibility condition for the energy thus reduces to

$$2\epsilon = l^2 \tag{10}$$

and we introduce the standard index $N = |l| + 1$.

This gives the energy values (8) for the case $n = 1$. The dimensions of the irreducible representations of $SO(2)$ are of course known to be equal to 1, and as there are two representations (values of J) corresponding to the eigenvalue

$$\epsilon_N = (N-1)^2, \qquad N = 1, 2, \ldots \tag{11}$$

we have

$$\begin{aligned} m_N &= 2 \qquad \text{for all} \quad N > 1, \\ m_1 &= 1. \end{aligned} \tag{12}$$

Of course the double degeneracy of the spectrum corresponds topologically to the fact that for $\Phi > 0$ the orbit space Q_0 consists of two points.

Remark: The abstract one-dimensional H-atom model is of definite interest for such areas as the theories of excitons[3], atoms[4], and interaction of electrons with the surface of liquid helium[1], just to mention a few of them. Using momentum representation it was analyzed by Yepez et al.[6], while Davtyan et al.[2] and Boya et al.[17] verify that Moser's equivalence[18] holds also in dimension one and explained some of the peculiarities of the direct solution of the quantum-mechanical problem.[5]

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