HOLOMORPHIC REPRESENTATIONS AND COHERENT STATES

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Abstract

We call a Lie group G admissible if its Lie algebra $\mathfrak{g}$ has the property that the extension $\mathfrak{g} \oplus \mathbb{R}$ contains pointed generating invariant cones. This class contains all compact groups, all simple hermitean groups, certain solvable groups and also mixed groups such as the semidirect product of the Heisenberg group and the symplectic group. In this note we describe an interesting class of irreducible unitary representations of such groups which can be characterized by three properties: they admit a holomorphic extension to some complex semigroup, the associated Lie algebra module is a highest weight module, and the representation is a coherent state representation. If the representation under consideration is assumed to have discrete kernel, then the admissibility of the group G follows from the existence of such a representation.

1. HOLOMORPHIC REPRESENTATIONS

1.1. Ol'shanskiĭ Semigroups

A closed convex cone W in the Lie algebra $\mathfrak{g}$ is called invariant if it is invariant under the adjoint action. Given a Lie algebra $\mathfrak{g}$, a generating invariant convex cone $W \subseteq \mathfrak{g}$, and a discrete central subgroup D of the simply connected group corresponding to the Lie algebra $\mathfrak{g} + i(W \cap (-W))$ which is invariant under complex conjugation, there exists a semigroup $S = \Gamma(\mathfrak{g}, W, D)$ called the *Ol'shanskiĭ semigroup* defined by this data.[1] Here $D \cong \pi_1(S)$ is the fundamental group of S. The semigroup S contains a dense open semigroup ideal called the *interior* S^0 which is a complex manifold on which the semigroup multiplication is holomorphic. Complex conjugation on the cone $\mathfrak{g} + iW$ integrates to an antiautomorphism $s \mapsto s^*$ of S which is antiholomorphic on S^0. The Ol'shanskiĭ semigroups are the domains for the holomorphic extensions of unitary representations. We note that if $W = \mathfrak{g}$, then $\Gamma(\mathfrak{g}, W, D)$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ which admits a complex conjugation.

Definition 1.1. Let us write $B(\mathcal{H})$ for the C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. A *holomorphic representation* of a Ol'shanskiĭ semigroup S is a weakly continuous monoid morphism $\pi: S \to B(\mathcal{H})$ such that π is holomorphic on the complex manifold S^0 and $\pi(s)^* = \pi(s^*)$ holds for all $s \in S$. One can think of

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