A COVARIANT APPROACH TO THE KUSTAANHEIMO–STIEFEL BUNDLE IN THE MAGNETIC MONOPOLE THEORY

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Abstract

A new covariant description of the magnetic monopole based on the consideration of a whole set of isomorphic bundles is proposed.

1. INTRODUCTION

As is well known the Hopf bundle $S^3 \to S^2$ or its extension $\dot{R}^4\backslash\{0\} \to \dot{R}^3 \backslash\{0\}$ – the so-called Kustaanheimo–Stiefel bundle (KSB) – are the most adequate mathematical constructions for description of abelian static magnetic monopole.[1] The magnetic monopole (the Dirac monopole) was introduced in quantum mechanics by Dirac in 1931.[2] In the same year Hopf constructed his bundle.[3] But the Dirac monopole was described in bundle terminology only in 1975.[4–5] (Previous treatment of the monopole has used a potential singular along the Dirac string.[6]) The consideration of the monopole field as a gauge field in the bundle made it possible.

Note that the Dirac monopole is a physical object with nontrivial topology. This means that the corresponding KSB is a nontrivial one. It is interesting to note that the hydrogen atom – a physical object with trivial topology – was described in KSB before than the magnetic monopole was. However such description has no gauge character.

Further we recall the following important property of the Dirac potential, namely, that the Dirac potential is a generalized symmetrical gauge field, i.e. it is invariant under the rotation group up to a gauge transformation.[7] Therefore it is natural for the Dirac potential with an arbitrary line singularity to use such a construction, which is transformed into an isomorphic one under the rotation group. As we will see, this means the consideration of a whole set of isomorphic bundles. In this sense we will speak about a new covariant approach.

2. COVARIANT KST

Definition 1 – The surjective quadratic map $\pi : \dot{R}^4 \equiv R^4\backslash\{0\} \to \dot{R}^3 \equiv R^3\backslash\{0\}$ defined by

$$\tilde{x}_1 = 2(x_1x_3 + x_2x_4),\ \tilde{x}_2 = 2(x_2x_3 - x_1x_4),\ \tilde{x}_3 = {x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2,\ x \neq 0, \quad (1)$$

where $(x_j) \equiv x \in R^4$ and $(\tilde{x}_\alpha) \equiv \tilde{x} = \pi(x)$, $x \neq 0$ $(i, j, ... = 1, ..., 4;\ \alpha, \beta, ... = 1, 2, 3)$ is called the Kustaanheimo–Stiefel transformation (KST).[8–9]

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