AFFINE POISSON STRUCTURES IN ANALYTICAL MECHANICS

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Abstract

If the space-time is a product of the space and the time the Poisson structure on the phase bundle is used to describe dynamics of mechanical systems. Further it is shown that if the space-time is a fibration over the time, then the Poisson structure has to be replaced by an affine Poisson structure.

1. TIME-DEPENDENT SYSTEMS

1.1. Time Independent Systems

In order to define a time-independent system the space-time has to be the product of space and time represented by the real line $\mathbb{R}$. For a time-independent system with configuration manifold Q, the infinitesimal dynamics is a submanifold D of $\mathsf{T}\mathsf{T}^*Q$. In particular cases D is the image of a vector field. The cotangent bundle T^*Q with the canonical 2-form ω_Q is a symplectic manifold.[1-3] The tangent bundle $\mathsf{T}\mathsf{T}^*Q$ of the cotangent bundle with the tangent 2-form $d_\mathsf{T}\omega_Q$ is a symplectic manifold as well.[4,5] We say that the system is *Lagrangian* if the dynamics D is a Lagrange submanifold of $(\mathsf{T}\mathsf{T}^*Q, d_\mathsf{T}\omega_Q)$.

Let us denote by τ_Q the canonical projection $\tau_Q\colon \mathsf{T}Q \to Q$ and by π_Q the canonical projection $\pi_Q\colon \mathsf{T}^*Q \to Q$. There are three, fundamental for the analytical mechanics, isomorphisms of vector bundles:

$$\kappa_Q\colon (\tau_{\mathsf{T}Q}\colon \mathsf{T}\mathsf{T}Q \to \mathsf{T}Q) \longrightarrow (\mathsf{T}\tau_Q\colon \mathsf{T}\mathsf{T}Q \to \mathsf{T}Q) \tag{1.1}$$

$$\alpha_Q\colon (\mathsf{T}\pi_Q\colon \mathsf{T}\mathsf{T}^*Q \to \mathsf{T}Q) \longrightarrow (\pi_{\mathsf{T}Q}\colon \mathsf{T}^*\mathsf{T}Q \to \mathsf{T}Q) \tag{1.2}$$

$$\beta_Q\colon (\mathsf{T}\pi_Q\colon \mathsf{T}\mathsf{T}^*Q \to \mathsf{T}Q) \longrightarrow (\pi_{\mathsf{T}^*Q}\mathsf{T}^*\mathsf{T}^*Q \to \mathsf{T}^*Q) \tag{1.3}$$

The mapping α_Q is also a symplectomorphism of $(\mathsf{T}\mathsf{T}^*Q, \mathsf{T}\pi_Q)$ and $(\mathsf{T}^*\mathsf{T}Q, \pi_{\mathsf{T}Q})$. The mapping β_Q is a symplectomorphism of $(\mathsf{T}\mathsf{T}^*Q, \mathsf{T}\pi_Q)$ and $(\mathsf{T}^*\mathsf{T}^*Q, \pi_{\mathsf{T}^*Q})$.

Let the dynamics D of a system be a Lagrangian submanifold of $(\mathsf{T}\mathsf{T}^*Q, \mathsf{T}\pi_Q)$. It follows that $\alpha_Q(D)$ and $\beta_Q(D)$ are Lagrangian submanifolds of $(\mathsf{T}^*\mathsf{T}Q, \pi_{\mathsf{T}Q})$ and $(\mathsf{T}^*\mathsf{T}^*Q, \pi_{\mathsf{T}^*Q})$ respectively. By a theorem of Hörmander $\alpha_Q(D)$ and $\beta_Q(D)$ can be