Lie Group Analysis of the Willmore and Membrane Shape Equations

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Abstract The present paper is concerned with the geometric Lie symmetry groups of the Willmore and shape equations—the Euler-Lagrange equations associated with the Willmore and Helfrich functionals. The ten-parameter group of special conformal transformations in the three-dimensional Euclidean space, which in known to be the symmetry group of the Willmore functional, is recognized as the largest group of geometric transformations admitted by these equations in Monge representation. The conserved currents of ten linearly independent conservation laws, which correspond to the variational symmetries of the Willmore equation and hold on its smooth solutions, are derived. The shape equation is found to admit only a six-parameter subgroup of the aforementioned ten-parameter group. Each symmetry admitted by the shape equation is its variational symmetry as well and the corresponding conserved currents are obtained.

1 Introduction

A wide variety of objects exhibit elastic behaviour in ordinary operation, and "thin" or "thin-walled" ones are of special interest concerning various human activities. In many cases their equilibrium shapes are formed due to bending. The problem for determination of the equilibrium shapes of such objects (bars, rods, archs, rings, pipes, baloons, etc.) is usually reduced to analysis of geometric objects—curves and surfaces [6]. Such an analysis is based on two concepts—extrema of the curvature

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